

Graphs of kei and their diameters

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Abstract

A kei on $[n]$ can be thought of as a set of maps $(f_x)_{x \in [n]}$, where each f_x is an involution on $[n]$ such that $(x)f_x = x$ for all x and $f_{(x)f_y} = f_y f_x f_y$ for all x and y . We can think of kei as loopless, edge-coloured multigraphs on $[n]$ where we have an edge of colour y between x and z if and only if $(x)f_y = z$; in this paper we show that any component of diameter d in such a graph must have at least 2^d vertices and contain at least 2^{d-1} edges of the same colour. We also show that these bounds are tight for each value of d .

1 Introduction

A *kei* (or *involutive quandle*) is a pair (X, \triangleright) , where X is a non-empty set and $\triangleright : X \times X \rightarrow X$ is a binary operation such that:

1. For any $y, z \in X$, there exists $x \in X$ such that $z = x \triangleright y$;
2. Whenever we have $x, y, z \in X$ such that $x \triangleright y = z \triangleright y$, then $x = z$;
3. For any $x, y, z \in X$, $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$;
4. For any $x \in X$, $x \triangleright x = x$;
5. For any $x, y \in X$, $(x \triangleright y) \triangleright y = x$.

Note that conditions 1 and 2 above are equivalent to the statement that for each y , the map $x \mapsto x \triangleright y$ is a bijection on X .

A *quandle* is a pair (X, \triangleright) satisfying conditions 1–4 above, while a *rack* is a pair (X, \triangleright) satisfying conditions 1–3. As mentioned in [2], racks originally developed from correspondence between J.H. Conway and G.C. Wraith in 1959, while quandles were introduced independently by Joyce [4] and Matveev [5] in 1982 as invariants of knots, and kei were first studied by Takasaki [8] in 1943. Fenn and Rourke [3] provide a history of racks and quandles, while Nelson [6] gives an

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overview of how these structures relate to other areas of mathematics; a recent paper by Stanovský [7] gives a thorough survey of the history of research on kei.

As a first example of a kei, note that for any set X , if we define $x \triangleright y = x$ for all $x, y \in X$, we obtain a kei known as the *trivial kei* T_X . Let G be a group and let X be the set of all involutions of G . If we define a binary operation $\triangleright : X \times X \rightarrow X$ by $x \triangleright y := y^{-1}xy$, then (X, \triangleright) is a kei; it is an example of a *conjugation quandle*. For a further example, define a binary operation on $[n]$ by setting $i \triangleright j := 2j - i \pmod n$; $([n], \triangleright)$ is known as a *dihedral kei*.

For any kei (X, \triangleright) , we can define a set of involutions $(f_y)_{y \in X}$ by setting $(x)f_y = x \triangleright y$ for all x and y . The following well-known result (see for example, [3], [2]) gives the correct conditions for a collection of maps $(f_y)_{y \in X}$ to define a kei.

Proposition 1.1. *Let X be a set and $(f_x)_{x \in X}$ be a collection of functions each with domain and co-domain X . Define a binary operation $\triangleright : X \times X \rightarrow X$ by $x \triangleright y := (x)f_y$. Then (X, \triangleright) is a kei if and only if f_y is an involution for each $y \in X$ and the following conditions hold: for all $y, z \in X$ we have*

$$f_{(y)f_z} = f_z f_y f_z, \quad (1.1)$$

and for all $x \in X$ we have

$$(x)f_x = x. \quad (1.2)$$

Proof. As noted earlier, each f_y is a bijection; it remains to show that items 3 and 4 in the definition of a kei are equivalent to (1.1) and (1.2) respectively, while item 5 is equivalent to the statement that each f_y is an involution. This is essentially a reworking of the definition and we omit the simple details. \square

This means that we can just as well define a kei on a set X by the set of maps $(f_y)_{y \in X}$, providing they are all involutions satisfying (1.1) and (1.2). We will move freely between the two definitions, with $x \triangleright y = (x)f_y$ for all $x, y \in X$ unless otherwise stated.

Now observe that any kei on X can be represented by a multigraph on X ; we give each vertex a colour and then put an edge of colour i from vertex j to vertex k if and only if $(j)f_i = k$. This is well-defined as each f_i is an involution, so $(j)f_i = k$ if and only if $(k)f_i = j$. We then remove all loops from the graph; i.e. if $(j)f_i = j$ we don't have an edge of colour i incident to j .

It will be helpful to recast the representation of kei by multigraphs in a slightly different setting. Let V be a finite set and let $\sigma \in \text{Sym}(V)$ be an involution; then we can define a simple graph G_σ on V by letting $uv \in E(G_\sigma)$ if and only if $u \neq v$ and $(u)\sigma = v$. As σ is a disjoint product of transpositions, we see that G_σ consists of a partial matching and some isolated vertices. We can now extend this definition to the case of multiple involutions in a natural way.

Definition 1.2. Suppose $\Sigma = \{\sigma_1, \dots, \sigma_k\} \subseteq \text{Sym}(V)$ is a set of involutions on a set V . Define a loopless multigraph $G_\Sigma = (V, E)$ with a k -edge-colouring by putting an edge of colour i from u to v if and only if $u \neq v$ and $(u)\sigma_i = v$.

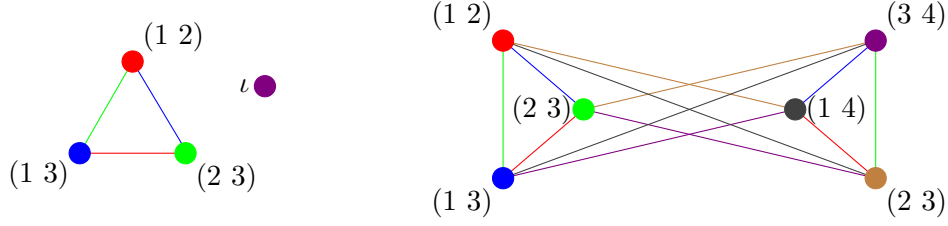


Figure 1: Graphical representations of two kei. Both of these are subquandles of conjugation quandles; S_3 on the left and S_4 on the right.

We also define the *reduced* graph G_Σ^0 to be the simple graph on V obtained by setting $e = uv \in E(G_\Sigma^0)$ if and only if there is at least one edge from u to v in G_Σ .

Observe that if $\Sigma' \subseteq \Sigma$, then $G_{\Sigma'}$ is a subgraph of G_Σ . Now let us return specifically to kei.

Definition 1.3. Let $K = (X, \triangleright)$ be a kei, and let $(f_y)_{y \in X}$ be the associated maps. For any $S \subseteq X$, define $\Sigma_S = \{f_y \mid y \in S\}$. Then by G_S we mean the multigraph G_{Σ_S} in the sense of Definition 1.2; G_S thus has an associated $|S|$ -edge-colouring, although if $|S| = 1$ we may not necessarily consider G_S as being coloured. We will also write $G_K = G_X$, indicating the graph for the whole kei.

Figure 1 gives two examples of graphs representing kei. Before stating our main result, we will need some more definitions. Let (X, \triangleright) be a kei; then a *subkei* of (X, \triangleright) is a kei $(Y, \triangleright|_{Y \times Y})^1$ where $Y \subseteq X$. Thus a subset $Y \subseteq X$ forms a subkei if and only if for all $y, z \in Y$, $(z)f_y \in Y$. For any $T \subseteq X$ and $u, v \in X$, denote by $d_T(u, v)$ the graph distance between u and v in the graph G_T . As G_T^0 is the simple graph on X formed by ignoring all colours and multiple edges, $d_T(u, v)$ is clearly identical to the graph distance between u and v in the reduced graph G_T^0 . We will prove the following result.

Theorem 1.4. Let (X, \triangleright) be a kei, and let (S, \triangleright) be a subkei. Let $C \subseteq X$ span a component of G_S (or equivalently of G_S^0) and suppose that $G_S[C]$ has diameter d . Then $|C| \geq 2^d$, and further there exists some $k \in S$ such that there are at least 2^{d-1} edges of colour k in $G_S[C]$.

The remainder of the paper is organised as follows. In Section 2 we show how to construct a large number of shortest paths between two vertices u and v that are connected in G_S . In Section 3 we will show that any sequence of colours occurring in order (from u to v) on a shortest path corresponds to a different vertex connected to u , which will prove the result. In Section 4 we give examples to show that Theorem 1.4 is tight for all values of d .

¹The notation $\triangleright|_{Y \times Y}$ in the above context will always be abbreviated to \triangleright , with the restriction to the subset Y left implicit.

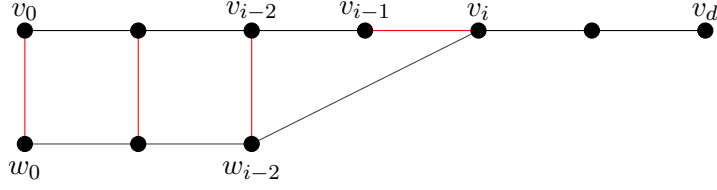


Figure 2: An example with $i = 4$. The colour c_i is represented by red, with the colours on the lower path being the colours c' , c'' etc. found as we progress from right to left along the upper path.

2 Shortest paths

We begin by showing that there are many shortest paths between pairs of vertices.

Lemma 2.1. *Let $S \subseteq X$ be a subkei, and let $u, v \in X$ such that $d_S(u, v) = d \in [2, \infty)$; let $u = v_0 v_1 \cdots v_d = v$ be a path of length d in G_S between u and v . For each i , let c_i be the colour of the $v_{i-1} v_i$ edge on the path, so $(v_i) f_{c_i} = v_{i-1}$. Now fix some $i \geq 2$ and define $w_k = (v_k) f_{c_i}$ for $k = 0, \dots, i-2$. Then the vertices $w_0, \dots, w_{i-2}, v_0, \dots, v_d$ are all distinct and $v_0 w_0 \cdots w_{i-2} v_i \cdots v_d$ is a (shortest) uv -path in G_S .*

Proof. (See Figure 2 throughout). For any $j \leq i-2$, denote by $P(j)$ the statement that the vertices $w_j, \dots, w_{i-2}, v_0, \dots, v_d$ are all distinct, and that the path $v_0 \cdots v_j w_j \cdots w_{i-2} v_i \cdots v_d$ is a (shortest) uv -path in G_S . Then the lemma is the statement $P(0)$; we will prove that $P(j)$ holds for all j by reverse induction.

Firstly, as $w_{i-2} = (v_{i-2}) f_{c_i}$ and $v_i = (v_{i-1}) f_{c_i}$ we see that $w_{i-2} \notin \{v_{i-1}, v_i\}$ (or there would be a vertex with two incident edges of the same colour). If we had $w_{i-2} = v_l$ for some $l > i$ then $v_0 \cdots v_{i-2} v_l \cdots v_d$ would be a uv -path in G_S of length $d - (l - i + 1) < d$, a contradiction. Now observe that

$$w_{i-2} := (v_{i-2}) f_{c_i} = (v_{i-1}) f_{c_{i-1}} f_{c_i} = (v_i) f_{c_i} f_{c_{i-1}} f_{c_i} = (v_i) f_{c'},$$

where $c' = (c_{i-1}) f_{c_i}$; note that $c' \in S$ as S is a subkei and so there is an edge in G_S between v_i and w_{i-2} . So now suppose $w_{i-2} = v_l$ for $l < i-1$; then $v_0 \cdots v_l = w_{i-2} v_i \cdots v_d$ is a uv -path in G_S of length $d - (i - l - 1)$, again a contradiction. So $w_{i-2} \neq v_l$ for any l , and we can use the $v_{i-2} w_{i-2}$ and $w_{i-2} v_i$ edges, of colours c_i and c' respectively, to construct a uv -path $v_0 \cdots v_{i-2} w_{i-2} v_i \cdots v_d$ of length d in G_S . This establishes $P(i-2)$.

So now take $j < i-2$ and suppose $P(j+1)$ holds. As $v_{j+1} w_{j+1}, \dots, v_{i-2} w_{i-2}$ and $v_{i-1} v_i$ are all edges of colour c_i , $w_j \neq v_l$ for $j < l \leq i$ and it is also not equal to any other w_k . As before, if $w_j = v_l$ for $l > i$ we can construct a uv -path in G_S of length $d - (l - i + 1)$, a contradiction. Now observe that

$$w_j := (v_j) f_{c_i} = (v_{j+1}) f_{c_{j+1}} f_{c_i} = (w_{j+1}) f_{c_i} f_{c_{j+1}} f_{c_i} = (w_{j+1}) f_{c''},$$

where $c'' = (c_{j+1})f_{c_i} \in S$, so there is an edge in G_S between w_j and w_{j+1} . Now suppose $w_j = v_l$ for some $l \leq j$; then $v_0 \cdots v_l = w_j w_{j+1} \cdots w_{i-2} v_i \cdots v_d$ is a uv -path in G_S of length $d - (j - l + 1)$, a contradiction (note the existence of the $w_{j+1} v_i$ -path follows from the assumed $P(j+1)$). Hence $w_j \neq v_l$ for any l , and we can use the $v_j w_j$ and $w_j w_{j+1}$ edges, of colours c_i and c'' respectively, to construct a uv -path $v_0 \cdots v_j w_j \cdots w_{i-2} v_i \cdots v_d$ of length d in G_S . Thus $P(j)$ holds, and so by reverse induction $P(0)$ holds, proving the lemma. \square

Corollary 2.2. *Let P be any shortest uv -path in G_S . Then no two edges of P have the same colour.*

Proof. Suppose there is a path $u = v_0 \cdots v_d = v$ and a colour $c \in S$ such that $(v_{j-1})f_c = v_j$ and $(v_{i-1})f_c = v_i$ for some $j < i - 1$ (we can't have $j = i - 1$ as then v_j is incident to two edges of colour c). From the lemma, with $c = c_i$, $w_j = (v_j)f_{c_i} \neq v_l$ for any l , but here $w_j = (v_j)f_c = v_{j-1}$, a contradiction. \square

3 Sequences of elements

Suppose we are considering a shortest path $P : (u = v_0, \dots, v_d = v)$ in G_S . In the light of Corollary 2.2, and to ease notation, we will assume without loss of generality that $X = [n]$ and $[d] \subseteq S$, and that the edge between v_{i-1} and v_i in P is of colour i . Now for each strictly increasing sequence $\mathbf{s} = (a_1, \dots, a_r)$ of elements from $[d]$, define

$$u_{\mathbf{s}} = (u)f_{a_1} \cdots f_{a_r},$$

where we note $u_{\mathbf{s}} \in S$ as S is a subkei. Now define $U_0 = \{u\}$ and for $r = 1, \dots, d$, define

$$U_r = \{u_{\mathbf{s}} \mid \mathbf{s} \text{ is strictly increasing, } |\mathbf{s}| = r\},$$

so in particular $U_d = \{v\}$. Now let $\mathbf{e}_i = (1, \dots, i)$ for all i ; then $v_i = (u)f_1 \cdots f_i = u_{\mathbf{e}_i}$ and so $v_i \in U_i$ for all i . We also have the property that as \mathbf{s} is increasing, $a_i \geq i$ for any i , with equality if and only if the subsequence consisting of the first i terms of \mathbf{s} is the canonical sequence \mathbf{e}_i .

We will show that any strictly increasing sequence \mathbf{s} can appear at the start of a shortest uv -path, and that any such path passes sequentially through each of U_0, U_1, \dots, U_d .

Lemma 3.1. *Let $S \subseteq X$ be a subkei, and let $u, v \in X$ be such that $d_S(u, v) = d \in [2, \infty)$; let $u = v_0 v_1 \cdots v_d = v$ be a path of length d in G_S between u and v , where the $v_{i-1} v_i$ edge is of colour i for each i . Let $\mathbf{s} = (a_1, \dots, a_r)$ be a strictly increasing sequence of elements from $[d]$. Then there is a shortest path $P_{\mathbf{s}} : (u = x_0 x_1 \cdots x_{a_r-1} x_{a_r} = v_{a_r} \cdots v_d)$ such that $x_k \in U_k$ for $1 \leq k < a_r$ and the $x_{k-1} x_k$ edge in $P_{\mathbf{s}}$ is of colour a_k for $1 \leq k \leq r$ (which means in particular that $x_r = u_{\mathbf{s}}$).*

Proof. We shall prove the following stronger statement by induction on r : there exists a shortest path $P_s : (u = x_0 x_1 \cdots x_{a_r-1} x_{a_r} = v_{a_r} \cdots v_d)$ such that for $1 \leq k < a_r$,

$$x_k = u_{\mathbf{t}} \quad (3.1)$$

for a (strictly increasing) sequence \mathbf{t} of length k whose largest element is at most a_r , and that for $1 \leq k \leq r$ the $x_{k-1} x_k$ edge in P_s is of colour a_k . These statements clearly imply the result.

First consider the base case $r = 1$; if $\mathbf{s} = \mathbf{e}_1 = (1)$ then the original path P suffices as the first edge is of colour 1, so $v_1 = (u)f_1 = u_{(1)}$. Hence we may assume $a_1 > 1$. Then applying Lemma 2.1 to P with $i = a_1$, there exists a shortest path $P' : (u = v_0 w_0 \cdots w_{a_1-2} v_{a_1} \cdots v_d)$ in G_S , with an edge of colour a_1 between v_k and w_k for $0 \leq k \leq a_1 - 2$; in particular, the first edge of P' is of colour a_1 . Now for $k \leq a_1 - 2$, $w_k = (v_k)f_{a_1} = (u_{\mathbf{e}_k})f_{a_1}$, so $w_k = u_{(\mathbf{e}_k, a_1)} \in U_{k+1}$. So we obtain (3.1) by setting $x_k = w_{k-1}$ for $1 \leq k < a_1$ and putting $P_s = P'$.

So now take $r > 1$ and assume the result for smaller r . If $a_r = r$ we have $\mathbf{s} = \mathbf{e}_r$, and in this situation the original path P suffices as the first r edges are of colours $1, \dots, r$ and $v_k \in U_k$ for $1 \leq k \leq r$. So we can assume that $a_r > r$.

By applying the inductive hypothesis to the sequence $\mathbf{s}' = (a_1, \dots, a_{r-1})$ we see that there exists a path

$$P_{\mathbf{s}'} : (u = y_0 y_1 \cdots y_{a_{r-1}-1} y_{a_{r-1}} = v_{a_{r-1}} \cdots v_d)$$

such that, for $1 \leq k < a_{r-1}$, $y_k = u_{\mathbf{t}}$ for a strictly increasing sequence \mathbf{t} whose largest element is at most a_{r-1} . We also have that the $y_{k-1} y_k$ edge is of colour a_k for $1 \leq k < r$, but we don't know the colours of the edges $y_{k-1} y_k$ for $r \leq k \leq a_{r-1}$. But as $a_r > a_{r-1}$ the $v_{a_{r-1}} v_{a_r}$ edge of colour a_r is still present in $P_{\mathbf{s}'}$; hence we can apply Lemma 2.1 to $P_{\mathbf{s}'}$, with $i = a_r$, to obtain a shortest path $P'' : (u = v_0 w_0 \cdots w_{a_r-2} v_{a_r} \cdots v_d)$ in G_S , with an edge of colour a_r between y_k and w_k for $0 \leq k \leq \min\{a_{r-1}, a_r - 2\}$ (note that $a_r - 2 < a_{r-1}$ if and only if $a_r = a_{r-1} + 1$). We aim to show that for all $0 \leq k \leq a_r - 2$, $w_k = u_{\mathbf{t}'}$ for some sequence \mathbf{t}' of length $k + 1$ whose largest element is at most a_r . The proof will vary slightly depending on whether $a_r = a_{r-1} + 1$ (see Figure 3) or $a_r > a_{r-1} + 1$ (see Figure 4).

Fix a k such that $0 \leq k \leq a_{r-1} - 1 \leq a_r - 2$; by the inductive hypothesis (specifically (3.1)) $y_k = u_{\mathbf{t}}$ for some sequence \mathbf{t} whose k th and largest element is at most a_{r-1} . But $w_k = (y_k)f_{a_r} = (u_{\mathbf{t}})f_{a_r}$, so $w_k = u_{(\mathbf{t}, a_r)}$; as $a_r > a_{r-1}$ we have $w_k \in U_{k+1}$, where $w_k = u_{\mathbf{t}'}$ for some sequence \mathbf{t}' whose largest element is at most a_r . If $a_r = a_{r-1} + 1$ then we have considered all the vertices w_1, \dots, w_{a_r-2} .

Now suppose that $a_r > a_{r-1} + 1$; in this case we also have an edge of colour a_r between v_k and w_k for $a_{r-1} \leq k \leq a_r - 2$. But as before, $w_k = (v_k)f_{a_r} = (u_{\mathbf{e}_k})f_{a_r} = u_{(\mathbf{e}_k, a_r)}$ for any such k , so $w_k \in U_{k+1}$ and $w_k = u_{\mathbf{t}'}$ for some sequence \mathbf{t}' whose largest element is at most a_r . Thus we have the desired result on w_k for the entire range $0 \leq k \leq a_r - 2$.

Now as $a_r > r$ and $a_{r-1} \geq r - 1$ we have $\min\{a_{r-1}, a_r - 2\} \geq r - 1$, so there is always an edge of colour a_r between y_{r-1} and w_{r-1} . Consider the path P'''

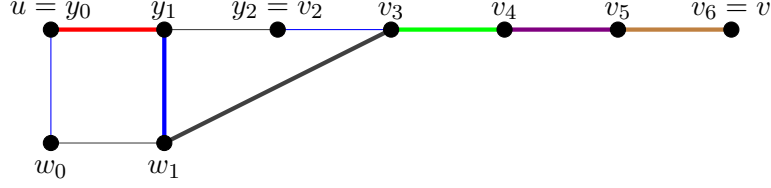


Figure 3: Suppose $\mathbf{s} = (2, 3)$ and that edges of colour 2 are red while those of colour 3 are blue. The top path $P_{\mathbf{s}'}$ corresponds to the sequence (2), and the bottom path P'' is a result of applying Lemma 2.1. The new path P''' is shown in bold.

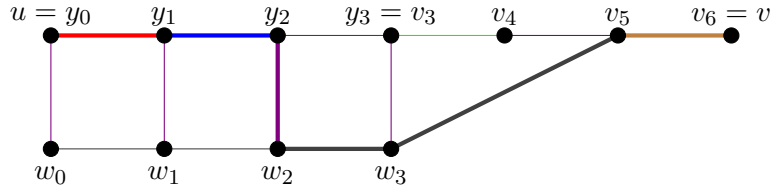


Figure 4: Suppose we now have the sequence $\mathbf{s} = (2, 3, 5)$, with edges of colour 5 being violet. The top path $P_{\mathbf{s}'}$ is the (relabelled) bold path from Figure 3, while the bottom path is again a result of applying Lemma 2.1; note the additional edges. The new path P''' is shown in bold.

obtained by using the first $r - 1$ edges of $P_{\mathbf{s}'}$, the edge $y_{r-1}w_{r-1}$ and then the remaining edges of P'' ; we obtain

$$P''' : (uy_1 \cdots y_{r-1}w_{r-1} \cdots w_{a_r-2}v_{a_r} \cdots v_d),$$

where the first r edges are of colours a_1, \dots, a_r . We get the result by putting $x_k = y_k \in U_k$ for $1 \leq k < r$ and $x_k = w_{k-1} \in U_k$ for $r \leq k < a_r$, and setting $P_{\mathbf{s}} = P'''$. \square

We can now use this result to show that different sequences correspond to distinct vertices.

Corollary 3.2. *Let \mathbf{s} and \mathbf{t} be distinct, strictly increasing sequences from $[d]$. Then $u_{\mathbf{s}} \neq u_{\mathbf{t}}$.*

Proof. Let $|\mathbf{s}| = q$ and $|\mathbf{t}| = r$, where we may assume $q \leq r$, and suppose that $u_{\mathbf{s}} = u_{\mathbf{t}}$. Write $\mathbf{s} = (a_1, \dots, a_q)$ and $\mathbf{t} = (b_1, \dots, b_r)$ and consider the paths $P_{\mathbf{s}} : (ux_1 \cdots x_{a_q} = v_{a_q} \cdots v_d)$ and $P_{\mathbf{t}} : (uy_1 \cdots y_{b_r} = v_{b_r} \cdots v_d)$ as described in Lemma 3.1; note that $x_q = u_{\mathbf{s}} = u_{\mathbf{t}} = y_r$. Thus we may replace the uy_r -segment of $P_{\mathbf{t}}$ with the ux_q -segment of $P_{\mathbf{s}}$ to obtain a uv -walk in $G_{\mathcal{S}}$ of length $d - (r - q)$, a contradiction if $q < r$. This shows that $U_q \cap U_r = \emptyset$ for $q \neq r$.

So suppose $q = r$ (so $x_r = y_r$) and for now that $a_r \neq b_r$; we may assume that $a_r < b_r$. Now replace the ux_r -segment of $P_{\mathbf{s}}$ with the uy_r -segment of $P_{\mathbf{t}}$ to obtain

a new uv -path

$$P' : (uy_1 \cdots y_r = x_r \cdots x_{a_r-1} x_{a_r} = v_{a_r} \cdots v_{b_r-1} v_{b_r} \cdots v_d),$$

where we note this is a path as both P_s and P_t pass sequentially through the pairwise disjoint sets U_0, U_1, \dots, U_d . But both the $y_{r-1}y_r$ and $v_{b_r-1}v_{b_r}$ edges on P' are of colour b_r , contradicting Corollary 2.2.

Now note that in any case $x_{r-1} = (x_r)f_{a_r}$ and $y_{r-1} = (y_r)f_{b_r}$, so if $a_r = b_r$ we have $x_{r-1} = y_{r-1}$. An easy inductive argument (and the fact that $s \neq t$) shows that there exists some $p \leq r$ such that $x_i = y_i$ for $p \leq i \leq r$ but $a_p \neq b_p$, so we may apply the above argument to the sequences $s' = (a_1, \dots, a_p)$ and $t' = (b_1, \dots, b_p)$ to get a contradiction, thus proving the result. \square

This allows us to prove the main result.

Proof of Theorem 1.4. If $d = 0$ or 1 then the result is trivial, so suppose $d \geq 2$. $G_S[C]$ has diameter d , so there exist $u, v \in C$ such that $d_S(u, v) = d$; as before, assume that $X = [n]$ and $[d] \subseteq S$, and that there is a shortest uv -path using edges of colours $1, \dots, d$ sequentially. There are 2^d strictly increasing sequences of elements from $[d]$ (including the empty sequence) and from Corollary 3.2 these correspond to 2^d distinct vertices (including u). Hence $|C| \geq 2^d$.

Note that there are 2^{d-1} strictly increasing sequences from $[d-1]$; adding d to the end of such a sequence s' gives a strictly increasing sequence s from $[d]$. But then $(u_{s'})f_d = u_s$ so the edge $u_{s'}u_s$ is of colour d for each such sequence s' ; as all vertices are distinct this gives us 2^{d-1} edges of colour d . \square

4 Extremal examples

In this section we construct a family of extremal examples, one for each d . Let $X_d := \{u_1, \dots, u_d\} \cup \{0, 1\}^d$ be a set of $2^d + d$ elements, and for each $v \in \{0, 1\}^d$ and $S \subseteq [d]$, define v_S to be the element of $\{0, 1\}^d$ obtained by changing only the coordinates in S , i.e.

$$\{i \in [d] \mid (v_S)_i \neq v_i\} = S.$$

We will now define a set of bijections on X_d ; for each $1 \leq i \leq d$, set $(u_j)f_{u_i} = u_j$ for all j and $(v)f_{u_i} = v_{\{i\}}$ for all $v \in \{0, 1\}^d$. We will also set $f_v = \iota$ for all $v \in \{0, 1\}^d$.

Note that $(v_{\{i\}})_{\{i\}} = v$ and thus each map is an involution; as we also have $(x)f_x = x$ for each $x \in X_d$, we need only show that (1.1) holds to prove that we have defined a kei K . Note that for any distinct $i, j \in [d]$ and $v \in \{0, 1\}^d$,

$$(v)f_{u_i}f_{u_j} = (v_{\{i\}})f_{u_j} = v_{\{i,j\}} = (v_{\{j\}})f_{u_i} = (v)f_{u_j}f_{u_i},$$

and it follows that all the maps $(f_x)_{x \in X_d}$ commute. Hence (1.1) reduces to the statement that $f_{(y)f_z} = f_y$ for all y, z ; this is easily seen to be true.

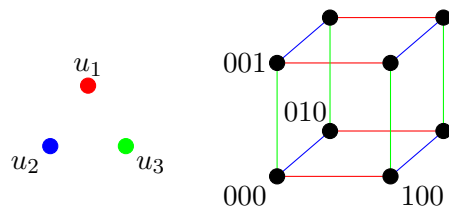


Figure 5: An example of a kei on X_3 .

Now consider the graph G_K ; by construction, G_K consists of d isolated vertices and a coloured copy of the cube Q_d (see Figure 5). The cube has diameter d , size 2^d and contains 2^{d-1} edges of each colour u_1, \dots, u_d .

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